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Planar graphs are 1-relaxed, 4-choosable

William Cushing [a], H.A. Kierstead [b]

[a] *Department of Computer Science, Arizona State University, Tempe, AZ 85287, USA*

[b] *Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287, USA*

A R T I C L E I N F O

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A B S T R A C T

We show that every planar graph $G = (V, E)$ is 1-relaxed, 4-choosable. This means that, for every list assignment L that assigns a set of at least four colors to each vertex, there exists a coloring f such that $f(v) \in L(v)$ for every vertex $v \in V$ and each color class $f^{-1}(\alpha)$ of f induces a subgraph with maximum degree at most 1.

1. Introduction

List coloring was introduced by Erdős, Rubin and Taylor [3] and Vizing [6]. For a graph $G = (V, E)$ and a set of colors $\mathcal{C}$, a *list assignment* is a function L on V that assigns a set $L(v) \subseteq \mathcal{C}$ of *usable* colors to every vertex $v \in V$. An *L-coloring* f is a proper coloring of G such that $f(v) \in L(v)$ for every vertex $v \in V$. A list assignment L is a *k-list assignment* if each list has size at least k. A graph G is *k-choosable* if G is L-colorable for every k-list assignment L. Thomassen [5] proved that every planar graph is 5-choosable and Voigt [7] constructed a planar graph that is not 4-choosable.

Cowen, Cowen and Woodall [1] considered relaxed coloring of planar graphs. An *r-relaxed* coloring of G is a coloring such that each color class induces a graph with maximum degree at most r. So a proper coloring is a 0-relaxed coloring. We say that f is a *(k, r)-coloring* if f is an r-relaxed coloring that uses k colors; a graph is *(k, r)-colorable* if it has a (k, r)-coloring. Cowen et al. proved that every planar graph is $(3, 2)$-colorable. This bound is best possible; let G be a planar graph consisting of a path P together with two additional vertices x, y that are both adjacent to all vertices of P and each other. It is easy to see that any $(3, 1)$-coloring or $(2, r)$-coloring of G must put x and y in the same color class, provided that P is sufficiently long. Thus if we identify the vertex x in $\max(2, r + 1)$ copies of G, we obtain a planar graph that is neither $(3, 1)$-colorable nor $(2, r)$-colorable.

Eaton and Hull [2] and independently Škrekovski [4] combined the notions of list coloring and relaxed coloring. A graph G is *(k, r)-choosable* if G has an r-relaxed L-coloring for every k-list assignment L. They used an extension of Thomassen's proof that every planar graph is 5-choosable to

E-mail addresses: wcushing@asu.edu (W. Cushing), kierstead@asu.edu (H.A. Kierstead).

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show that every planar graph is (3, 2)-choosable. The purpose of this paper is to prove the following theorem, which was the last remaining question left open in [2,4].

Theorem 1. *Every planar graph is* (4, 1)*-choosable.*

Most of our notation is standard; possible exceptions include the following. For an integer n, we set $[n] := \{1, \dots, n\}$; so $[0] = \emptyset$. We denote the set of non-negative integers as $\mathbb{N}$; so $0 \in \mathbb{N}$. For a set A and element x we set $A + x := A \cup \{x\}$ and $A - x := A \setminus \{x\}$; so $[n] + 0 = \{0, 1, \dots, n\}$. For graphs G and H, $G \pm H$, $G \pm A$, and $G \pm x$ are the usual super/subgraphs; let $x \in G$ denote $x \in V(G)$ for x a vertex and $x \in E(G)$ for x an edge.

For vertex sets X and Y, let $E(X, Y) := E(G[X], G[Y])$ denote the edges between X and Y; for graphs H and H', let $E(H, H') := (E(G) \setminus E(H)) \setminus E(H')$. A *separation* of G is a pair (X, Y) such that $G = G[X] + G[Y]$; so $E(X, Y) = \emptyset$. A *proper* separation (X, Y) is such that $X \setminus Y$ and $Y \setminus X$ are non-empty; so $X \cap Y$ is a cutset. We will also say that (H, H') is a (proper) separation if $(V(H), V(H'))$ is a (proper) separation, especially if H and H' are induced subgraphs ($H = G[V(H)]$ and $H' = G[V(H')]$). We denote the *image* of a function f on A in the usual way (so $f(A) := \{f(x) : x \in A\}$); except that for the *neighborhood* function, N, it is more convenient to define $N(A)$, for a vertex set A, as $N(A) := (\bigcup_{v \in A} N(v)) \setminus A$. We also set $N(H) := N(V(H))$ for a graph H. The effect is that $N(A)$ isolates A from the rest of G, just as the neighborhood of a single vertex does. That is, given $V' \subseteq V(G)$, $N(V') = (\bigcup_{v \in V'} N(v)) \setminus V'$ is the middle of a, possibly improper, separation $(V' \cup N(V'), V(G) \setminus V')$.

Let G be a plane graph. Any cycle Q in G separates the plane into an *interior* region and an *exterior* region; let $\mathrm{Int}[Q]$ and $\mathrm{Ext}[Q]$ denote the maximum subgraphs of G contained, respectively, in those regions—including Q. So $(\mathrm{Int}[Q], \mathrm{Ext}[Q])$ is a separation of G. A *triangle* is a 3-cycle, and G is *nearly triangulated* if all its faces, except possibly its exterior face, are bounded by triangles. A *separating triangle* Q is a triangle such that $(\mathrm{Int}[Q], \mathrm{Ext}[Q])$ is a proper separation.

Let $P = v_1 \dots v_n$ be a path. Then $v_i P$ denotes the subpath $v_i \dots v_n$, $P v_j$ denotes the subpath $v_1 \dots v_j$, and $v_i P v_j$ denotes the subpath $v_i \dots v_j$ if $i \le j$ and $v_j \dots v_i$ otherwise. Let $\mathring{P}$ be the subpath formed by the internal vertices of P, $\mathring{P} := P - \{v_1, v_n\} = v_2 \dots v_{n-1}$; similarly for $x, y \in P$, $\mathring{x}P := xP - x$, $P\mathring{y} := Py - y$, and $\mathring{x}P\mathring{y} := xPy - \{x, y\}$. Allow concatenation of this notation; for example, $PxQyR$ denotes the walk up to x in P, from x to y in Q, and from y to the end of R. Also let these same subpaths be defined with respect to walks; for our purposes it is enough to treat a walk as its longest initial subpath. So for $W = v_1 P v_n v_1 \dots$ the path $v_n W v_1$ is $v_n v_{n-1} \dots v_1$ since the longest initial subpath of W is P.

2. Set-up

Our methods extend the techniques introduced by Thomassen and Eaton and Hull. In order to make our induction arguments work we use a more general formulation of relaxed list coloring. Fix a graph $G = (V, E)$, a set of colors $\mathcal{C}$ and a number $r \in \mathbb{N}$.

A *relaxed coloring* is a function $f : V \to \mathcal{C}$. A *flaw* is an edge xy such that $f(x) = f(y)$; if $f(x) = f(y) = \alpha$ then xy is an *α-flaw*. We say that x is *flawless* if it is not incident to a flaw. So f is an r-relaxed coloring if v is incident to at most r flaws for all vertices $v \in V$. A *generalized list assignment* (hereafter shortened to *list assignment*) is a function L on V such that $L(v)$ is a sequence $(L^0(v), \dots, L^r(v))$ of disjoint subsets of $\mathcal{C}$. Let $L^*(v) := \bigcup_{i=0}^{r} L^i(v)$. An L-coloring is a relaxed coloring f such that, for all $v \in V$, $f(v) \in L^*(v)$ and, for the unique i such that $f(v) \in L^i(v)$, v is incident to at most i flaws. The *capacity* $|L|$ of L is (also) a function on V, such that $|L|(v)$ is the sequence $(|L^0(v)|, \dots, |L^r(v)|)$.

Since we are currently only interested in 1-relaxed colorings, we now simplify the notation by assuming that $r = 1$. Then each vertex is associated with two sets of colors: $L^0(v)$ and $L^1(v)$. Moving a color α from $L^1(v)$ to $L^0(v)$ places a stronger requirement on its use. We refer to this as *protecting* α at v. Completely eliminating α from $L^*(v)$ also strengthens the requirement of L; we refer to this operation as *removing* α at v. We allow redundant applications of these operations: protecting α at v does nothing if $\alpha \notin L^1(v)$; similarly, removing α at v does nothing if $\alpha \notin L^*(v)$. We say that L' is a *restriction* of L (on X) if it can be obtained from L by repeatedly protecting and/or removing colors (only at vertices in X: $L'(v) = L(v)$ for $v \notin X$). Observe that the capacity of a list assignment with $r = 1$

is a function into $\mathbb{N}^2$. Let $(a, b), (a', b') \in \mathbb{N}^2$. We say that (a, b) is a *restriction* of (a', b'), denoted as $(a, b) \preceq (a', b')$, if $a + b \leq a' + b'$ and $b \leq b'$. Similarly, if $w, w' : V \to \mathbb{N}^2$ then we say that w is a *restriction* of w', also denoted by $w \preceq w'$, if $w(v) \preceq w'(v)$ for all vertices v. Finally, observe that if L' is a restriction of L, then the capacity of L' is a restriction of the capacity of L: $L' \preceq L \Rightarrow |L'| \preceq |L|$.

Our plan is to prove Theorem 1 by proving the following technical statement by induction. This is similar to the approach of Thomassen and Eaton and Hull. Indeed, the only substantial difference is in the case where G has no separating triangles and is 3-connected. Note that the last vertex of the boundary, b_n, plays a special role in the complicated hypothesis: the three cases (A, B, C) differ only in constraints placed upon b_n.

Theorem 2. *Let $G = (V, E)$ be a 2-connected nearly triangulated plane graph with exterior face boundary $B = b_1 b_2 \ldots b_n b_1$. Let L be a list assignment for G such that*

$$L^*(b_1) = L^i(b_1) = \{\alpha\},$$
$$L^*(b_2) = L^i(b_2) = \{\beta\},$$
$$i = 1 \Leftrightarrow \alpha = \beta,$$
$$\text{and} \quad |L|(v) = \begin{cases} (1, 2) & \text{if } v \in V(B) \setminus \{b_n, b_1, b_2\} \\ (0, 4) & \text{if } v \in V \setminus V(B). \end{cases}$$

If either

(A) *$n = 3$ and $|L|(b_n) = (1, 0)$ and $\alpha, \beta \notin L^*(b_n)$, or*
(B) *$|L|(b_n) = (1, 2)$, or*
(C) *$|L|(b_n) \in \{(2, 0), (0, 1)\}$ and $\alpha \notin L^*(b_n)$ and $N(b_1) \cap N(b_n) \cap V(B) = \emptyset$,*

then G is L-colorable.

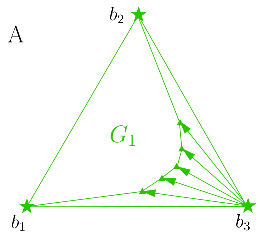
We shall say that (G, L) *satisfies* *A if there exists $L' \preceq L$ such that the hypothesis of the theorem holds, by Case A, for (G, L').[1] We assign similar meanings to *B and *C.

We first show that Theorem 2 implies Theorem 1. Let G be any planar graph and let L be any list assignment for G with $|L|(v) \succeq (0, 4)$ for all vertices v. Adding vertices and edges to G only makes it harder to color, so we may assume that it is edge maximal with at least three vertices (i.e., a triangulation). Then $(\tilde{G}, L)$ satisfies *A for any plane drawing $\tilde{G}$ of G. So by Theorem 2, $\tilde{G}$ is colorable for some restriction of L, and thus also L-colorable. Therefore, G is L-colorable.

3. Easy cases

In this section we begin the proof of Theorem 2. We do the base step and the easy cases of the induction. In the next section we state and prove complex technical lemmas in support of the hard case. These lemmas carry out the lion's share of work; leaving us with only small details to take care of in the final section. The reader may prefer to postpone the proofs of these lemmas until after understanding their use in the last section. The important cases are illustrated in Fig. 1.

Let (G, L) satisfy the hypothesis of the theorem (by Cases A, B, or C). We argue by induction on $|G|$. The base step $|G| = 3$ is trivial, since *A must hold, so we may assume Case A holds, and then a forced L-coloring of $B = G$ exists. So consider the induction step.



Case A. Let $\gamma \in L^0(b_3)$ and $Q := N(b_3) \setminus \{b_1, b_2\}$. Each vertex of Q has capacity $(0, 4)$. Let L_1 be the list assignment for $G_1 := G - b_3$ obtained by removing γ at each vertex of Q, and thus at worst reducing the capacity to $(0, 3) \succeq (1, 2)$. As all vertices of Q are on the outer boundary of G_1, (G_1, L_1) satisfies *B. By the induction hypothesis, G_1 has an L_1-coloring f. By hypothesis, $\gamma \notin f(\{b_1, b_2\}) = \{\alpha, \beta\}$, and so, by the choice of L_1, $\gamma \notin f(N(b_3))$. Therefore, coloring b_3 with γ extends f to an L-coloring of G.

[1] We could instead write the theorem using $\succeq$. Equalities serve to simplify the case analysis.

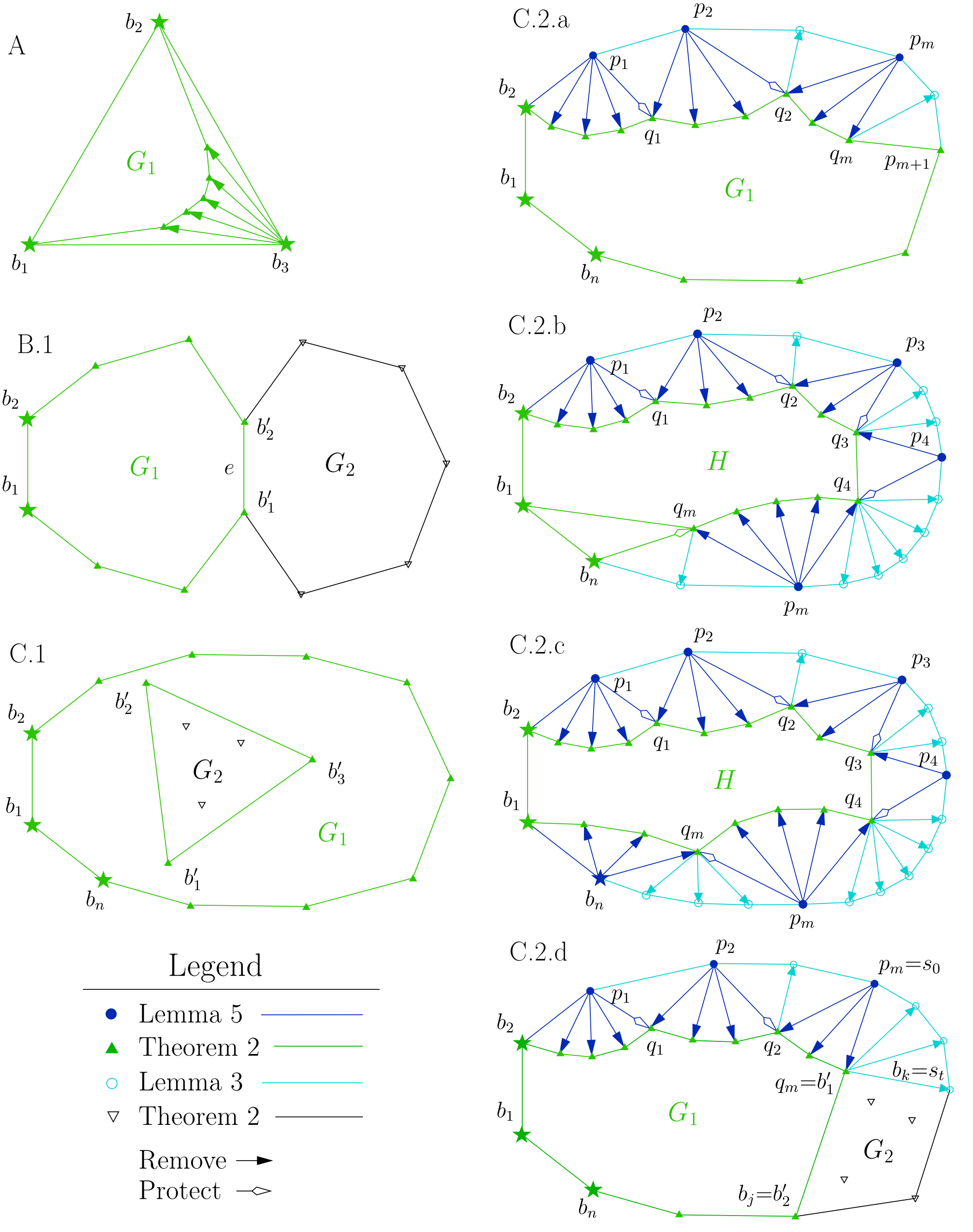
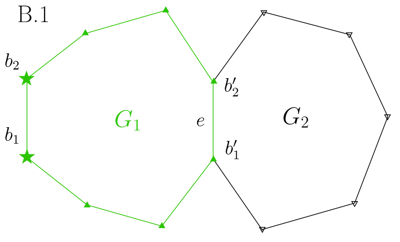


Fig. 1. Case Diagrams. G is colored in the order of the legend. "Theorem 2" means coloring either by induction or by the case itself, e.g., C.2.b colors b_n directly and H by induction. G is undirected; arrows designate restriction operations.

Case B.0. B is chordless. If $n = 3$ then (G, L) also satisfies *A, and so we argue by Case A instead. Otherwise ($n > 3$), (G, L) also satisfies *C, and so we argue by Case C instead. Note that, by altering the induction hypothesis, one could eliminate this subcase. However, that only complicates applying the induction hypothesis in the rest of the proof.



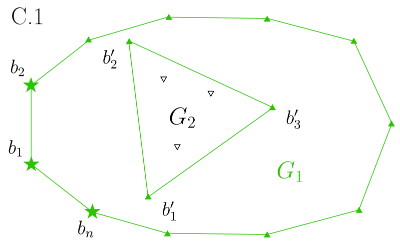
Case B.1. B has a chord $e := b'_2b'_1$. Then e lies on two unique cycles Q_1 and Q_2 spanning the area of G, with $b_1b_2 \in Q_1$ and $b_1b_2 \notin Q_2$. For $i \in [2]$, let $G_i = \mathrm{Int}[Q_i]$. Then (G_1, L) satisfies *B. By the induction hypothesis, G_1 has an L-coloring g. Let L_2 be the restriction of L obtained as follows, with $i \in [2]$. Remove every color except $g(b'_i)$ at b'_i, and if $g(b'_1) \neq g(b'_2)$ then protect $g(b'_i)$ at b'_i. In other words, $L_2(b'_i) := (\{g(b'_i)\}, \emptyset)$ unless e is a flaw in g; if so, $L_2(b'_i) := (\emptyset, \{g(b'_i)\})$. Observe that any L_2-coloring of G_2 agrees with g on the cutset and adds no further flaws to the cutset. Since (G_2, L_2) satisfies *B, the induction hypothesis yields an L_2-coloring h of G_2. It follows that $g \cup h$ is an L-coloring of G.

Case C.0. B has a chord $e = b'_2b'_1$. As in Case B.1, e lies on two unique cycles Q_1 and Q_2 spanning the area of G, with $b_1b_2 \in Q_1$ and $b_1b_2 \notin Q_2$. For $i \in [2]$, let $G_i = \mathrm{Int}[Q_i]$.

Case C.0.a. $b_n \in Q_1$. Then (G_1, L) satisfies *C. By the induction hypothesis, G_1 has an L-coloring g. As demonstrated previously (in Case B.1), there exists a restriction L_2 of L on the endpoints of e such that L_2-colorings of G_2 agree with g without adding new flaws to the endpoints of e and (G_2, L_2) satisfies *B. Therefore g extends to an L-coloring of G.

Case C.0.b. $b_n \in Q_2 - Q_1$. Observe that the only possibility is that $b_1 = b'_1$, so $e = b_1b_i$ for some i such that $2 < i < n$. Therefore b_1 and b_n continue to be the first and last vertices of the boundary of G_2. Clearly they continue to have no common neighbors on that boundary.

Since (G_1, L) satisfies *B we have an L-coloring g of G_1. Again we can take a restriction L_2 of L on the endpoints of e that forces L_2-colorings of G_2 to agree with g without adding new flaws to those endpoints. Still as before $|L_2|$ is sufficiently large to satisfy the capacity-related constraints of the induction hypothesis, and so (G_2, L_2) satisfies *C by the observation above that the identities of b_1 and b_n remain unaltered. Therefore G is L-colorable.



Case C.1. G has a separating triangle $B' := b'_1b'_2b'_3b'_1$.[2]

Let $G_2 := \mathrm{Int}[B']$ and $G_1 := \mathrm{Ext}[B']$. Then (G_1, L) satisfies *C, and so by the induction hypothesis G_1 has an L-coloring g. Let L_2 be the restriction of L on B' as follows, for $i \in [3]$. Remove every color besides $g(b'_i)$ at b'_i. Unless b'_i has a flaw $e \in B'$, protect $g(b'_i)$ at b'_i. Then L_2-colorings of G_2 agree with g without adding new flaws to the vertices of B'; further, (G_2, L_2) satisfies *A.[3] Therefore an extension of g to an L-coloring of G exists.

So for the remainder, Case C.2 (see Section 5), we have that (C) holds, G is 3-connected (since B is chordless), and G has no separating triangles.

4. Lemmas

Consider (G, L) satisfying *C. One possibility is that the neighbors of the boundary, $N(B)$, induce another cycle B'. Intuitively this case should be easy to handle via induction. Were it further the case

2 Cases C.0 and C.1 are not mutually exclusive.

3 If necessary, rotate the names of the new boundary vertices.

that every vertex of B' had no more than two neighbors on B then it would, in fact, be relatively simple to continue. In that case we can flawlessly color $b_n \dots b_3$, remove them, and force colorings of the interior to complete those choices by taking an appropriate restriction on B'. As there would only be two exterior neighbors to be concerned about it would be possible to perform that restriction without violating the capacity bound, $(1, 2)$, of the induction hypothesis. However, in general, there will not always be only two neighbors from B' to B, nor will the neighbors of B form a cycle in the first place. Lemmas 3 and 5 address the former difficulty, and the arguments for Case C work around the latter difficulty.

In the course of our arguments we will often need to 2-color paths subject to various constraints and assumptions at the endpoints. So first observe that paths can be colored with many degrees of freedom in the relaxed setting:

Lemma 3. *Let $P = v_0 \dots v_t$ be a path, with $t \geq 1$. Let L be a list assignment such that $|L|(v_i) = (1, 1)$ for all $i \in [t-1]$.*

3.a *If $|L|(v_0) = (2, 0)$ and $|L|(v_t) = (1, 0)$ then P can be flawlessly L-colored.*
3.b *If $|L|(v_0) = (0, 1)$ and $|L|(v_t) = (0, 1)$ then P can be L-colored.*
3.c *If $|L|(v_0) = (0, 1)$, $|L|(v_t) = (1, 0)$ and $L^1(v_0) \cap L^0(v_1) = \emptyset$ then P can be L-colored.*
3.d *If $|L|(v_0) = (1, 0)$, $|L|(v_t) = (1, 0)$ and $L^0(v_0) \cap L^*(v_1) = \emptyset$ then P can be L-colored.*

We will invoke Lemma 3 on list assignments with excess capacity, as with Theorem 2. Note that if there exists $J' \preceq J$ such that the hypothesis of the lemma holds of (S, J') then the conclusion holds with respect to (S, J).

Proof. For (3.a) we can color $v_t \dots v_0$ flawlessly (by first-fit). For (3.d) we can color $v_t \dots v_1$ flawlessly, and then the forced choice at v_0 is not the color chosen for v_1. For (3.c) we can color $v_t \dots v_1$ flawlessly, and then the forced choice at v_0 is either different from the choice at v_1, or v_0v_1 is an α-flaw and by hypothesis we have $\alpha \in L^1(v_0) \cap L^1(v_1)$.

For (3.b) we argue by induction on t. Let $\delta \in L^1(v_0)$ and $\gamma \in L^1(v_1)$. For the base step, $t = 1$, the forced choices are an L-coloring of P; it makes no difference whether or not $\delta = \gamma$ as a flaw is allowed. So consider the induction step $t > 1$. If $\delta \notin L^0(v_1)$ then (3.c) applies and we are done. So $\delta \in L^0(v_1)$, and thus $\delta \neq \gamma$. Obtain a restriction M of L by removing δ at v_1. Then $|M|(v_1) = (0, 1)$, and the induction hypothesis yields an M-coloring of v_1P. Since v_1 must be colored γ, coloring v_0 with δ completes an L-coloring of P. ■

Notation. Let $P = v_0 \dots v_k$ be a path and $q \notin V(P)$ be a vertex. The structure obtained by joining q to every vertex of P, and distinguishing q, is called a *fan* and is denoted by $(P; q)$. A *trestle* is a structure $T = (P; p_1, \dots, p_{m+1}; Q; q_0, \dots, q_m)$ such that P is a p_1–p_{m+1} path with distinguished vertices p_i, $i \in [m+1]$, appearing in order (not necessarily consecutively) along P, Q is a q_0–q_m path with distinguished vertices q_i, $i \in [m] + 0$, appearing in order (also not necessarily consecutively) along Q, P and Q are disjoint, and if $m > 0$ then for all $i \in [m]$ both of the fans $(q_{i-1}Qq_i; p_i)$ and $(p_iPp_{i+1}; q_i)$ are subgraphs of T, otherwise $(m = 0)$, T is said to be *degenerate* and q_0p_1 is its only edge.

In other words, a trestle is a series of alternating fans. Fig. 2 depicts an example. Trestles and fans are simply graphs (satisfying certain constraints, with distinguished vertices), and so we reuse all notation defined for graphs with both trestles and fans. For example $V(T) := V(P) \cup V(Q)$ for any trestle $T = (P; p_1, \dots, p_{m+1}; Q; q_0, \dots, q_m)$. We argue by induction on trestles; so let all of the following be defined, for $0 \leq i \leq m$:

$$
\begin{aligned}
P_i &:= Pp_{i+1}, & \qquad P_i^- &:= P\mathring{p}_{i+1},\\
Q_i &:= Qq_i, & Q_i^+ &:= Qq_ip_{i+1},\\
T_i &:= (P_i; p_1, \dots, p_{i+1}; Q_i; q_0, \dots, q_i), \qquad \text{and} & T_i^- &:= T_i - E(Q_i^+).
\end{aligned}
$$

Consider (G, L) satisfying *C and suppose some $P \subset B$ and $Q = G[N(P)]$ induce a trestle T (see Fig. 1). We would like to color G, for some i, by combining an L-coloring g of $G - P_i^-$ obtained from the induction hypothesis with an L-coloring h of T_i^- obtained from Lemma 5. Of course g and h must

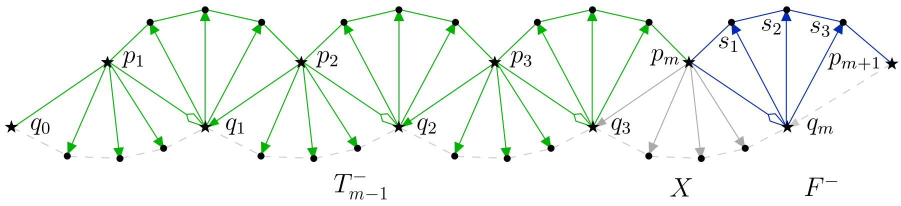


Fig. 2. An example trestle T along with our default coloring strategy. We partition the edges of T_m into the pieces Q_m^+ (the dashed edges), T^-_{m-1}, X, and F^-; the last three decompose T_m^-. Then we argue that we can compatibly color T_m^-, separately considering each of its pieces, given any coloring of Q_m^+ (subject to the restriction operations we perform). Open arrows represent protecting and solid arrows represent removing colors (at the destination).

agree on $N(P_i^-) = V(Q_i^+)$ (the cutset); we say that g and h are *compatible* if $g(v) = h(v)$ for all v where both colorings are defined.

However there is an added complication in the relaxed setting: for each $v \in Q_i^+$, we must ensure that v does not gain distinct flaws from g and h. We will manage flaws by developing a pair of related restrictions; for the purpose of discussion let g be a J-coloring of Q_i^+ and h a K-coloring of T_i^-. Note that we avoid double counting of flaws in $E(Q_i^+)$ by assigning them to g: h only colors $T_i^- = T_i - E(Q_i^+)$. Then we *a priori* decide, for each vertex of Q_i^+ and each color, which of the two colorings will be allowed to put a flaw of that color incident to that vertex. That is, if $\varepsilon \in J^1(q)$ for some $q \in Q_i^+$ then we will ensure that $\varepsilon \in K^0(q)$. Likewise, if $\lambda \in K^1(q)$ for some $q \in Q_i^+$ then we will ensure that $\lambda \in J^0(q)$.

To formalize this, we define the *L-complement* $\mathrm{Compl}_Z(L, M)$ of M on Z to be the restriction $\overline{M}$ of L on Z as follows:

$$\overline{M}^b(v) := \{\gamma : \exists a, \gamma \in L^{a+b}(v) \cap M^a(v)\} \quad \text{for } v \in Z, \tag{1}$$

$$\text{and} \quad \overline{M}(v) := L(v) \quad \text{for } v \notin Z. \tag{2}$$

Equivalently, since we are only working with at most 1 flaw ($r = 1$), for $v \in Z$:

$$\overline{M}^0(v) = (L^0(v) \cap M^0(v)) \cup (L^1(v) \cap M^1(v)),$$
$$\text{and} \quad \overline{M}^1(v) = L^1(v) \cap M^0(v).$$

To actually obtain $\overline{M}$ from L as a series of restriction operations, for each $v \in Z$, remove every $\alpha \notin M(v)$, remove every $\alpha \in M^1(v) \setminus L^0(v)$ (empty if $M \preceq L$), and protect every $\alpha \in M^1(v) \cap L^1(v)$. Alternatively, for the special case of M a restriction of L on Z then generate its complement simultaneously by performing each removal operation on both and protect every remaining color in exactly one of the two restrictions.

Then, as intended, vertices in Z cannot acquire more flaws from the union of an M-coloring with a $\mathrm{Compl}_Z(L, M)$-coloring than originally permitted by L. Formally:

Proposition 4. *For any list assignment L, restriction $M \preceq L$, and separation (X, Y) of a graph G, with $Z := X \cap Y$, the union of any M-coloring of $G[X]$ with any compatible $\mathrm{Compl}_Z(L, M)$-coloring of $G[Y] - E(G[Z])$ is an L-coloring of G.*

Proof. Let g M-color $G[X]$, h compatibly $\mathrm{Compl}_Z(L, M)$-color $G[Y] - E(G[Z])$, and [set] $f := g \cup h$. There is nothing to check for $v \notin Z$ by the assumption $M \preceq L$ (for $v \in X$) and (2) (for $v \in Y$). For $v \in Z$ and $\alpha = f(v)$, v is incident to at most a flaws in g and at most b flaws in h, where $\alpha \in M^a(v) \cap \mathrm{Compl}_Z(L, M)^b(v)$. Then, by (1), $\alpha \in L^{a+b}(v)$, and so each v is incident to at most the permitted number of flaws (and colored from its list). ■

With respect to the main argument (Case C.2; see Fig. 1), we color G by first finding an appropriate separation $(G - P_i^-, T_i)$ (with cutset $V(Q_i^+)$). Then we consult the following lemma to generate a restriction M of L (on Q_i^+) that allows the induction hypothesis to be applied to $(G - P_i^-, M)$. The arrows in the case diagrams summarize the restriction operations the lemma actually performs. The lemma guarantees that the remainder of the graph, T_i^-, can be compatibly colored by the complement of M, i.e. a compatible $\mathrm{Compl}_{Q_i^+}(L, M)$-coloring of T_i^- is guaranteed for every way of M-coloring $G - P_i^- \supseteq Q_i^+$. Then Proposition 4 finishes the argument; the union of the two colorings will L-color G.

The previous discussion accurately describes how we use Lemma 5 in the first, *shortcut* (SC), case of the lemma. The second, *trestle* (Tr), case provides restrictions with sufficient capacity in Q, but unlike (SC) requires that p_{m+1} be precolored (with δ). Besides that, our use of restrictions and their complements remains the same in the second case.

Lemma 5. *Let $T = (P; p_1, \dots, p_{m+1}; Q; q_0, \dots, q_m)$ be a trestle and L be a list assignment such that $|L^*(q_0)| = 1$, $|L|(v) = (0, 4)$ for all $v \in \mathring{q}_0 Q$, and $|L|(v) = (1, 2)$ for $v \in P\mathring{p}_{m+1}$. Either:*

(SC) *there exists $i \in [m]$ and a restriction M of L on $\mathring{q}_0 Q_i$ such that*

- (†a) *$|M|(v) \succeq (1, 2)$ for all $v \in \mathring{Q}_i$, and*
- (†b) *$|M|(q_i) = (0, 3)$, and*
- (†c) *$M(p_{i+1}) = L(p_{i+1})$ (this is vacuous, but contrasts with (‡c)), and*
- (†d) *every M-coloring of Q_i^+ is compatible with some $\overline{M}$-coloring of T_i^-, with $\overline{M} := \mathrm{Compl}_{Q_i^+}(L, M)$; or*

(Tr) *there exists a color $\pi \in \mathcal{C}$ and a mapping $\mathbf{M}$ from every $\delta \in L^1(p_{m+1}) \cup (L^0(p_{m+1}) - \pi)$ to a restriction M_δ of L on $\mathring{q}_0 Q_m^+$ such that*

- (‡a) *$|M_\delta|(v) \succeq (1, 2)$ for all $v \in \mathring{q}_0 Q_m$, and*
- (‡b) *if $m > 0$, or $\delta \neq \pi$, then $\delta \notin M_\delta^*(q_m)$, and*
- (‡c) *$M_\delta^*(p_{m+1}) = \{\delta\}$ and $\delta \in M_\delta^1(p_{m+1})$ iff $\delta \in L^1(p_{m+1}) - \pi$, and*
- (‡d) *every M_δ-coloring of Q_m^+ is compatible with some $\overline{M_\delta}$-coloring of T_m^-, with $\overline{M_\delta} := \mathrm{Compl}_{Q_m^+}(L, M)$.*

Notation. We interpret (SC) and (Tr) as defining predicates, and write $\mathrm{SC}_{T'}(i', M')$ to mean that case (SC) holds of trestle T' using i' and M' as the existentially guaranteed index and restriction, respectively. Similarly we write $\mathrm{Tr}_{T'}(\pi', \mathbf{M}')$ when case (Tr) holds of trestle T' using π' and $\mathbf{M}'$ as the existentially guaranteed color and mapping, respectively. Note that fixing $\mathbf{M}'$ by (Tr) further defines a restriction M'_λ for every choice of color λ. We will also abbreviate this notation whenever one or more of these objects is clear in context or its identity is not required. For example, we write $\mathrm{SC}(i)$ to mean that the lemma holds via the shortcut case for some unspecified trestle, with index i and unspecified restriction.

Proof. We argue by induction on m. First consider the base step $m = 0$. So T is the edge $q_0 p_1$. We claim (Tr) holds. Let π be the only color in $L^*(q_0)$, and for any color δ form M_δ by restricting L at p_1 so that (‡c) holds. Then (‡a) holds vacuously and (‡b) holds because $M_\delta^*(q_0) = L^*(q_0) = \{\pi\}$. Finally (‡d) holds trivially, since T_m^- has no edges.[4]

Now consider the induction step $m > 0$. Assume that $\mathrm{SC}(i)$ does not hold for any $i < m$, as otherwise we are done. Then by the induction hypothesis $\mathrm{Tr}_{T_{m-1}}(\pi', \mathbf{M}')$ holds. Recall that $|L|(p_m) = (1, 2)$ and that fixing $\mathbf{M}'$ further defines restrictions M'_λ of L, for each $\lambda \in L^1(p_m) \cup (L^0(p_m) - \pi')$. Let $\delta' \in L^1(p_m) - \pi'$ and $\gamma' \in L^*(p_m) \setminus \{\delta', \pi'\}$. We will, depending on the case, choose λ (the color of p_m) from $C := \{\delta', \gamma'\}$. Then $\lambda \neq \pi'$, and so:

$$\lambda \notin M'^*_\lambda(q_{m-1}) \quad \text{by (‡b)}, \tag{3}$$

$$M'^*_\lambda(p_m) = \{\lambda\} \quad \text{by (‡c), first part}, \tag{4}$$

$$\text{and} \quad \overline{M'_\lambda}(p_m) = (\{\lambda\}, \emptyset) \quad \text{by (‡c), second part}. \tag{5}$$

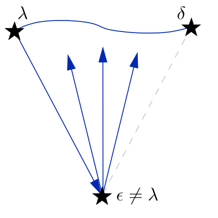
[4] If there are no M_δ-colorings, i.e. $\delta = \pi \in L^0(q_0)$ with $m = 0$, then (‡d) holds vacuously.

Let $S := s_0 \dots s_t := p_m P p_{m+1}$, $F := (S; q_m)$, $F^- := F - q_m p_{m+1}$, and X be the star with leaf set $V(q_{m-1} Q \mathring{q}_m)$ and root p_m (see Fig. 2). Notice that we can partition $E(T_m^-)$ as $\{E(T_{m-1}^-), E(X), E(F^-)\}$; also note that p_m is a cut vertex of T_m^-, separating F^- from $T_{m-1}^- + X$. For both $\lambda \in C$, obtain a restriction K_λ of L on $\mathring{Q}_m$ by setting $K_\lambda(v) := M'_\lambda(v)$ for each $v \in Q_{m-1}$, and removing λ at each leaf of X (which has no effect at q_{m-1} by (3)). Then K_λ satisfies (†a, ‡a) since M'_λ satisfies both with respect to T_{m-1}, and K_λ only alters lists in $\mathring{q}_{m-1} Q \mathring{q}_m$, leaving each with at least $(0, 3) \succeq (1, 2)$ capacity. Let $\overline{K_\lambda} := \mathrm{Compl}_{Q\mathring{q}_m}(L, K_\lambda)$.

Claim 6. *For both $\lambda \in C$, and every K_λ-coloring g' of $Q\mathring{q}_m$, there exists a compatible $\overline{K_\lambda}$-coloring h' of $T_{m-1}^- + X$ such that p_m is flawlessly colored λ in h'.*

Proof. Extend g' to an M'_λ-coloring g^* of Q_{m-1}^+ by coloring p_m with λ. By $\mathrm{Tr}_{T_{m-1}}(\pi', \mathbf{M}')$, there exists an $\overline{M'_\lambda}$-coloring h^* of T_{m-1}^- that is compatible with g^* and colors p_m flawlessly (by (5)). Since $\overline{K_\lambda}$ and $\overline{M'_\lambda}$ differ only at p_m (and $\lambda \in \overline{K_\lambda}^*(p_m) = L^*(p_m)$), h^* is also a $\overline{K_\lambda}$-coloring of T_{m-1}^-. By the choice of K_λ, $\lambda \notin K_\lambda^*(v)$ for any $v \in q_{m-1} Q \mathring{q}_m$. Thus $h' := g' \cup h^*$ is a compatible $\overline{K_\lambda}$-coloring of $T_{m-1}^- + X$ such that p_m is flawlessly colored λ. ■

Then the real work of the proof takes place on F^-. In each case below we will obtain a restriction, M or M_δ, by restricting K_λ on $\{q_m, p_{m+1}\}$ such that the capacity constraints hold. The new restriction does not restrict vertices in $T_{m-1}^- + X$ so the claim applies. Then to show (†d) or (‡d) it suffices to compatibly color F^- by the complement of the new restriction, forcing λ at p_m, since p_m is a cut vertex (flawlessly colored λ outside of F^-, by the claim).



Case 1. $|S| > 2$ and either (a) $\delta' \notin L^0(s_1)$ or (b) $\gamma' \notin L^1(s_1)$. If (a) choose $\lambda := \delta'$; if (b) choose $\lambda := \gamma'$. We claim that $\mathrm{SC}_T(m)$ holds. Let M be the restriction of K_λ (on q_m) obtained by removing λ at q_m (which satisfies (†b)). M satisfies (†a), since K_λ does, and (†c) is vacuous. So it suffices to prove (†d).

Consider any M-coloring g of Q_m^+; set $\delta := g(p_{m+1})$ and $\varepsilon := g(q_m)$. We must construct an $\overline{M}$-coloring of T_m^- that is compatible with g. By the claim, there is a compatible $\overline{M}$-coloring h' of $T_{m-1}^- + X$ that flawlessly colors p_m with λ. Since p_m is a cut vertex, and has no flaws yet, it suffices to extend h' to $h := h' \cup j$, where j is an $\overline{M}$-coloring of F^- with $j(p_m) := \lambda$, $j(p_{m+1}) := \delta$ and $j(q_m) := \varepsilon$. By the definition of $\overline{M}$, this amounts to coloring the vertices of $\mathring{S}$ from L while preventing flaws at q_m and p_{m+1}. In other words, we require j to be a J-coloring of S, where J is the restriction of L obtained as follows. Remove all colors except λ at p_m. Remove all colors except δ at p_{m+1}. Remove ε at every vertex of $\mathring{S}$ (thereby preventing flaws at q_m, since $\lambda \neq \varepsilon$); note that $|J|(v) \succeq (1, 1)$ for $v \in \mathring{S}$. Protect δ at p_{m+1} (thereby preventing flaws at p_{m+1}).

Case 1.a. $\delta' \notin L^0(s_1)$. So $\lambda = \delta'$. Then $\delta' \notin J^0(s_1)$. Since $\delta' \in L^1(p_m)$, it is also in $J^1(p_m)$. By Lemma 3.c, S has a J-coloring.

Case 1.b. $\delta' \in L^0(s_1)$ and $\gamma' \notin L^1(s_1)$. So $\lambda = \gamma'$. By hypothesis $|L^0(s_1)| = 1$ so we have $\gamma' \notin L^*(s_1) \supseteq J^*(s_1)$. By Lemma 3.d we have a J-coloring of S.

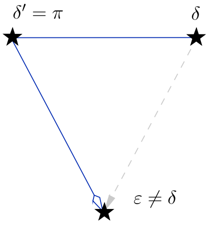
Case 2. $|S| = 2$ or $L(s_1) = (\{\delta'\}, \{\gamma', \sigma\})$, for some color σ. We claim that (Tr) holds. We will consider several subcases below. In each subcase we first choose π. Then we define $\mathbf{M}(\delta) := M_\delta$ for each $\delta \in L^1(p_{m+1}) \cup (L^0(p_{m+1}) - \pi)$. For each such δ we define M_δ by restricting K_λ on $\{q_m, p_{m+1}\}$ for some choice of λ, perhaps dependent on δ. Having chosen λ, we obtain M_δ by (i) removing δ at q_m [giving (‡b)], (ii) protecting λ at q_m, (iii) removing all colors except δ at p_{m+1} and (iv) protecting δ at p_{m+1} if $\delta = \pi$ [along with (iii), giving (‡c)]. Since $|K_\lambda|(q_m) = |L|(q_m) = (0, 4)$, (i), (ii) at worst give $|M_\delta|(q_m) = (1, 2)$, and, since $|K_\lambda|(v) \succeq (1, 2)$ for $v \in \mathring{q}_0 Q \mathring{q}_m$, M_δ satisfies (‡a). So it suffices to prove (‡d).

Consider an arbitrary M_δ-coloring g of Q_m^+ and let $\varepsilon := g(q_m)$; note that $\delta = g(p_{m+1})$ (by (iii)) and $\delta \neq \varepsilon$ (by (i)). By the claim, as in Case 1, there exists a compatible $\overline{M_\delta}$-coloring h' of $T_{m-1}^- + X$ with p_m flawlessly colored λ. Since p_m is a cut vertex of T_m^-, and colored flawlessly, it suffices to demonstrate a compatible $\overline{M_\delta}$-coloring j of F^-, as then $h' \cup j$ is an $\overline{M_\delta}$-coloring of T_m^-. The only uncolored vertices are in $\mathring{S}$, and for $v \in \mathring{S}$ we have $\overline{M_\delta}(v) = L(v)$ by definition (i.e. by (2)). For the remaining (colored) vertices:

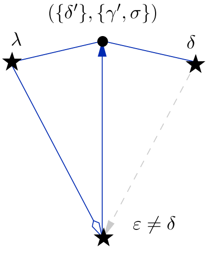
$$\text{if } \lambda = \delta' \text{ then } \lambda \in \overline{M_\delta}^1(p_m) = L^1(p_m) \quad \text{by (2)}, \tag{6}$$

$$\text{if } \varepsilon = \lambda \ \text{ then } \varepsilon \in \overline{M_\delta}^1(q_m) \quad \text{by (ii)}, \tag{7}$$

$$\text{and if } \delta = \pi \ \text{ then } \delta \in \overline{M_\delta}^1(p_{m+1}) \quad \text{by (iv)}. \tag{8}$$



Case 2.a. $|S| = 2$. So $p_m p_{m+1} \in E$ and $\mathring{S} = \emptyset$; thus F^- is the path $q_m p_m p_{m+1}$. Set $\pi := \delta'$, and regardless of δ, set $\lambda := \delta' = \pi$. Since $\delta \neq \varepsilon$, there is at most one flaw. If $p_m q_m$ is a flaw then $\varepsilon = \lambda$, and so is allowed by (6) and (7). If $p_m p_{m+1}$ is a flaw then $\delta = \lambda = \pi$, and so is allowed by (6) and (8).

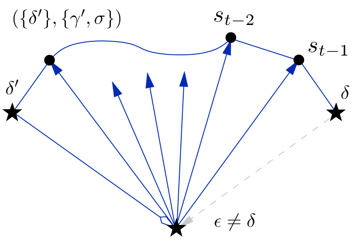


Case 2.b. $|S| = 3$ and $L(s_1) = (\{\delta'\}, \{\gamma', \sigma\})$. Set $\pi := \sigma$ and consider $\delta \in L^1(p_{m+1}) \cup (L^0(p_{m+1}) - \pi)$.

First suppose $\delta = \gamma'$. Set $\lambda := \gamma'$. Since $\varepsilon \neq \delta$, and $\lambda = \delta$, $p_m q_m$ is not a flaw. Moreover, there exists a safe color $\omega \in \overline{M_\delta^*}(s_1) \setminus \{\gamma', \varepsilon\} = L(s_1) \setminus \{\gamma', \varepsilon\}$ with which to color s_1.

Next suppose $\delta = \pi$. Set $\lambda := \delta'$. If $p_m q_m$ is a flaw then it is allowed by (6) and (7). For the remaining edges, observe that s_1 may be colored with $\pi = \sigma$, as this avoids a flaw at $s_1 p_m$ and $s_1 q_m$, and the last edge, $s_1 p_{m+1}$, is permitted to be a π-flaw by the case and (8).

Finally consider any $\delta \notin \{\gamma', \pi\}$. Set $\lambda := \delta'$. Then if $p_m q_m$ is a flaw it is allowed by (6) and (7). Coloring s_1 with $\omega \in \{\gamma', \sigma\} - \varepsilon$ introduces no flaws since $\delta, \lambda \notin \{\gamma', \sigma\}$.



Case 2.c. $|S| \geq 4$ and $L(s_1) = (\{\delta'\}, \{\gamma', \sigma\})$. Choose $\pi \in L^1(s_{t-1})$ so that, if possible, $\pi \notin L^1(s_{t-2})$. Set $\lambda := \delta'$, regardless of δ. We can allow ourselves to prevent all flaws in $E(q_m, \mathring{S})$, i.e., instead of compatibly $\overline{M_\delta}$-coloring F^-, it suffices to J-color $S' := q_m p_m S = F^- - E(q_m, \mathring{S})$ for the following restriction J of $\overline{M_\delta}$. Remove all colors except λ at p_m. Remove all colors except δ at p_{m+1}. Protect δ at p_{m+1}, unless $\delta = \pi$ (if so, then $\|J\|(p_{m+1}) = (0, 1)$ by (8)). Remove ε in $\mathring{S}$; note that $\|J\|(v) \succeq (1, 1)$ for $v \in \mathring{S}$. Since a δ'-flaw at $p_m s_1$ is impossible (by the case), any J-coloring of S immediately extends to a J-coloring of S': even if $\varepsilon = \delta'$, a δ'-flaw at $q_m p_m$ is permitted by (6) and (7). So it suffices to J-color S.

Case 2.c.i. $L^1(s_{t-1}) = L^1(s_{t-2})$. Suppose $\delta \notin L^1(s_{t-1})$. Pick $\omega \in J^1(s_{t-1})$ and note that $\omega \neq \delta$. Then by Lemma 3.b, Ss_{t-1} has a J-coloring with s_{t-1} colored ω (remove all colors but ω at s_{t-1} before applying the lemma). Clearly this extends to a J-coloring of S.

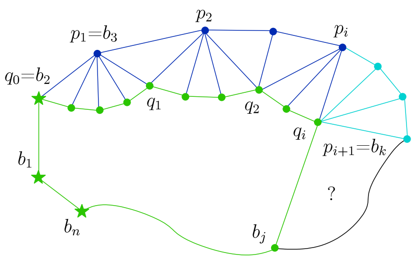
Otherwise $\delta \in L^1(s_{t-1}) = L^1(s_{t-2})$. Since $\varepsilon \neq \delta$, we also have that $\delta \in J^1(s_{t-2})$. By Lemma 3.b, Ss_{t-2} has a J-coloring with s_{t-2} colored δ (remove all other colors first). Only s_{t-1} remains; color it with $\omega \in J^*(s_{t-1}) - \delta$.

Case 2.c.ii. $L^1(s_{t-1}) \neq L^1(s_{t-2})$. So $\pi \in L^1(s_{t-1}) \setminus L^1(s_{t-2})$. Suppose $\varepsilon \neq \pi$: then $\pi \in J^1(s_{t-1})$. Pick $\omega \in J^1(s_{t-2})$ and note that $\omega \neq \pi$. Then, by Lemma 3.b, Ss_{t-2} has a J-coloring with s_{t-2} colored ω; clearly this extends to a J-coloring of Ss_{t-1} with s_{t-1} flawlessly colored π. This in turn extends to a J-coloring of S with p_{m+1} colored δ: either $\delta = \pi$ and the flaw is allowed, or $\delta \neq \pi$ and there is no flaw. Either way, S can be J-colored.

Otherwise $\varepsilon = \pi$, and so by the choice of π, $|J^1(s_{t-2})| = 2$. Then first-fit (Lemma 3.a) gives a J-coloring of $s_t s_{t-1} s_{t-2}$ with s_{t-2} flawlessly colored some color $\omega \in J^1(s_{t-2})$. By Lemma 3.b a J-coloring of Ss_{t-2} exists with s_{t-2} colored ω (possibly with a (permitted) flaw). So S is J-colorable. ■

5. No separating triangles, 3-connected

In this section we complete the proof of Theorem 2 by proving Case C.2. The goal is to find a trestle in the boundary (and its neighbors) satisfying (SC), i.e., a shortcut (Cases C.2.a and C.2.b). Failing that, we find a largest trestle satisfying (Tr). The latter arises either by including b_n in the trestle (Case C.2.c), or encountering a non-fan (Case C.2.d). In all cases we aim to reduce to *C by simply leaving b_n in the graph, but if b_n is part of the trestle then it and b_1 could acquire a common neighbor on the new boundary. If so then we treat b_n specially, and end up reducing to *B instead.



At stage 3

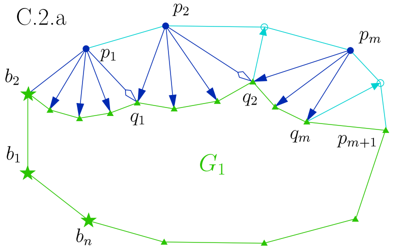
Case C.2. G is 3-connected and lacks separating triangles. We construct a trestle $T := (P; p_1, \ldots, p_{m+1}; Q; q_0, \ldots, q_m) \subset G$ in stages, T_i, beginning with T_0 (degenerate) and ending with T_m ($T = T_m$). P is an initial subpath of $B' := b_3 B b_n$, and Q is an initial subpath of the walk $b_2 \hat{B} b_1$, where $\hat{B}$ is the boundary of $G - B'$. Each stage i consists of first choosing two distinguished vertices q_i and p_{i+1}, and then extending Q through $b_2 \hat{B} b_1$ and P through B'. Initially, $i = 0$, $q_0 := b_2$ and $p_1 := b_3$.

Suppose that we have constructed T_{i-1} and consider stage i. Let $R := G[N(p_i)]$ and note that R is a path: R is Hamiltonian since G is nearly triangulated, and there are no shortcuts because G has no separating triangles. Let q_i be the first vertex on $\mathring{q}_{i-1} R$ with at least two neighbors (one is p_i) on B'. Let j be the largest index such that $b_j \in B'[N(q_i)]$ and k be the largest index such that the fan $(p_i B' b_k; q_i)$ is a subgraph of G. Set $p_{i+1} := b_k$ and $S := s_0 \ldots s_t := p_i B' p_{i+1}$; then $S \subseteq B'[N(q_i)]$ with equality if $k = j$. We will terminate if ever $k < j$, so $P_{i-1} = G[N(Q_{i-1})]$ and thus Q_{i-1} and R intersect only at q_{i-1}. So $Q_i := Q_{i-1} q_{i-1} R q_i = b_2 \hat{B} q_i$ is a path. Clearly $P_i := P_{i-1} p_i S p_{i+1} = b_3 B' p_{i+1}$ is a (disjoint) path. So $T_i := T_{i-1} + (q_{i-1} R q_i; p_i) + (p_i S p_{i+1}; q_i) \subseteq G[V(B') \cup N(B')] \subset G$ is well-defined.

If $k < j$, terminate in Case C.2.d (with $m := i$). Note that $p_m = p_{m+1}$ is possible, but only in this subcase. Otherwise, $k = j$, consult Lemma 5 with respect to (T_i, L).

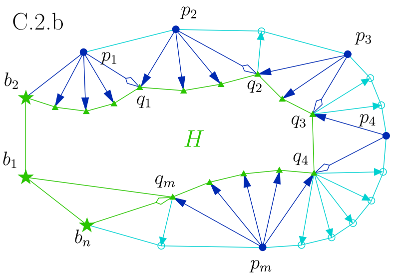
If SC(i, L_1) holds, set $m := i$, $B_1 := b_1 b_2 Q q_m p_{m+1} B b_n b_1$ and $G_1 := \mathrm{Int}[B_1] = G - P_m^-$; terminate in either Case C.2.a or Case C.2.b. Note that $|L_1|$ is large enough to apply the induction hypothesis to (G_1, L_1). Otherwise (Tr) holds.

If $j = n$ then terminate in Case C.2.c (with $m := i$). Failing all that (so $k = j$, (Tr) holds, and $j < n$), continue to the next stage.[5]



Case C.2.a. SC(m, L_1) holds and $N(b_1) \cap N(b_n) \cap V(B_1) = \emptyset$. Then (G_1, L_1) satisfies *C. By Lemma 5 any L_1-coloring of G_1 can be extended to an L-coloring of G; by the induction hypothesis such a coloring exists.

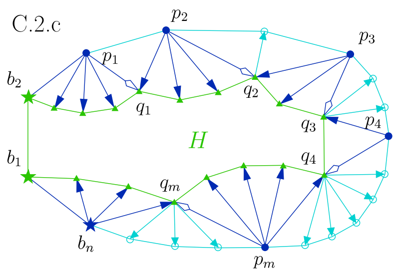
5 Note that $j < n$ cannot hold forever, i.e., the construction terminates.



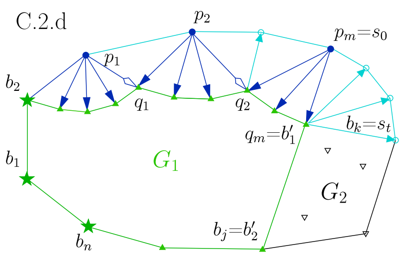
Case C.2.b. $SC(m, L_1)$ holds and $N(b_1) \cap N(b_n) \cap V(B_1) \neq \emptyset$. Any common neighbor must be on $\mathring{b}_2Q$ by (C), and since the first (and last) neighbor of b_n on Q is necessarily q_m we have $b_n = p_{m+1}$ and $b_1q_mb_nb_1 \subset G$. It still suffices to demonstrate an L_1-coloring of G_1, by Lemma 5, but we must work around the (non-separating) triangle $b_1q_mb_nb_1$ first. Let $H := G_1 - b_n$.

Suppose $\delta \in L^1(b_n)$ exists. Protect δ at q_m to obtain L_1'. Then $|L_1'|(q_m) \succeq (1, 2)$ by (†b) and so (H, L_1') satisfies *B. By the induction hypothesis, H has an L_1'-coloring f_1'. Extend f_1' to G_1 by coloring b_n (with δ). Either b_n is flawless and there is nothing to check, or the only possibility is that b_nq_m is a δ-flaw. Then by the choice of L_1', q_m is flawless in H, so the flaw is permitted. Either way, G_1 is L_1-colorable.

Otherwise, $|L^0(b_n)| = |L_1^0(b_n)| = 2$. More easily than above, H has an L_1-coloring f_1' that can be extended to an L_1-coloring of G_1 by coloring b_n with $\gamma \in L_1^0(b_n) - f_1'(q_m)$.



Case C.2.c. $j = n$ and $\mathrm{Tr}(\pi, \mathbf{M})$ holds. Let $R := G[N(b_n)] = G[N(p_{m+1})]$. Recall that $|L|(b_n) = (0, 1)$ or $|L|(b_n) = (2, 0)$. So a choice of $\delta \in L^1(b_n) \cup (L^0(b_n) - \pi)$ exists. Obtain a restriction L_1 of M_δ by removing δ at each vertex of q_mR. Note this has no effect on b_1 and q_m: (C) gives $\delta \notin L^*(b_1)$ directly, and the condition $N(b_1) \cap N(b_n) \cap V(B) = \emptyset$ implies $n > 3$, so $m > 0$ since $p_1 = b_3$ is not $p_{m+1} = b_n$, and thus (‡b) gives $\delta \notin M_\delta^*(q_m)$.[6] Let $H := G_1 - b_n$. Then (H, L_1) satisfies *B and so there exists an L_1-coloring f_1 of H by the induction hypothesis. By the choice of L_1, $\delta \notin f_1(N(b_n))$, so color b_n with δ, and note that G_1 is L_1-colorable. Therefore, by Lemma 5, G is L-colorable.



Case C.2.d. $k < j$. Let $B_1 := b_1b_2Qq_mb_jBb_nb_1$, $G_1 := \mathrm{Int}[B_1]$, $B_2 := q_mb_jBp_{m+1}q_m$, and $G_2 := \mathrm{Int}[B_2]$. Recall that $S = s_0 \dots s_t = p_mBb_k$ and $R = G[N(p_m)]$. Since the construction survived stage $m - 1$, $\mathrm{Tr}_{T_{m-1}}(\pi, \mathbf{M})$ holds. Let M_δ be the corresponding restriction of L for some $\delta \in L^1(p_m) - \pi$. Then by (‡b), since $\delta \neq \pi$, $\delta \notin M_\delta^*(q_{m-1})$. So remove δ at each vertex of $q_{m-1}Rq_m$ to obtain a restriction L_1 of M_δ.

The plan is to color G_1 by *C (or *B if $j = n$), T_{m-1} by Lemma 5, S by Lemma 3, and G_2 by *C. Note the reversal of direction in B_2; p_{m+1} takes on the role of the last boundary vertex and b_j takes on the role of the second boundary vertex.

As in Cases C.2.a and C.2.b we might have to work around b_n. If $N(b_1) \cap N(b_n) \cap V(B_1) = \emptyset$ then (G_1, L_1) satisfies *C, and so G_1 has an L_1-coloring. Otherwise b_1 and b_n have a common neighbor on Q and the only possibility is $j = n$ and $b_1q_mb_nb_1 \subset G$ is a non-separating triangle. If so, then we can remove b_n to obtain H, and further restrict L_1, obtaining L_1', by protecting $\delta' \in L^1(b_n)$ at q_m, or doing nothing if that choice does not exist. We have at most removed one color and protected one color at q_m so (H, L_1') satisfies *B. By the induction hypothesis H has an L_1'-coloring, and we can extend this to an L_1-coloring of G_1 by either picking some color in $L_1^*(b_n)$ different from the color of q_m, or if that is impossible then the δ'-flaw at b_nq_m is allowed by the choice of L_1'.

[6] Note that, unlike the other cases, we allow $\delta = \pi$. Still $\delta \notin M_\delta^*(q_m)$, but due to $m > 0$ instead of $\delta \neq \pi$.

In any case, G_1 possesses an L_1-coloring f_1. By Lemma 5 this coloring can be extended to the remainder of T_{m-1} with p_m flawlessly colored δ (since $\delta \in \overline{M_\delta}^0(p_m)$ and $\delta \notin f_1(R)$). That is, there exists an L-coloring f_1' of $G_1 + P_{m-1} + E(P_{m-1}, G_1) = \mathrm{Int}[b_1 B p_m q_m b_j B b_n b_1]$ with $f_1'(p_m) = \delta$ and no flaw incident to p_m. So S and G_2 remain.

Let $\alpha' := f_1'(q_m)$ and $\beta' := f_1'(b_j)$. Let L_2' be the restriction of L on B_2 obtained as follows. Remove every color besides α' at q_m and every color besides β' at b_j. If $\alpha' \neq \beta'$ then protect α' at q_m and β' at b_j. Remove every color besides δ at p_m, and remove α' at p_{m+1}. If $p_m = p_{m+1}$ then note that $\alpha' \neq \delta$. By the maximality of k, $N(q_m) \cap N(b_k) \cap V(B_2) = \emptyset$, and by the choice of L_2', $\alpha' \notin L_2'^*(p_{m+1})$. Therefore (G_2, L_2) satisfies *C for any restriction $L_2 \preceq L_2'$ (on p_{m+1}) leaving p_{m+1} with capacity $(2, 0)$ or $(0, 1)$ (at least).

Let J restrict L_2' by removing α' in S; note that $|J|(v) \succeq (1, 1)$ for $v \in \mathring{p}_m S$. Let $\sigma \in J^1(p_{m+1})$. Then by Lemma 3.b, S has a J-coloring j with p_{m+1} colored σ; if $p_m = p_{m+1}$ then the choice of $\sigma = \delta$ is forced.

Case C.2.d.i. $p_m = p_{m+1}$ or $s_{t-1}p_{m+1}$ is not a flaw in j. Then, even if $p_m = p_{m+1}$, p_{m+1} is flawlessly colored $\sigma \in J^1(p_{m+1})$ in $f_1' \cup j$. Let L_2 restrict L_2' by removing every color besides σ at p_{m+1}. By the induction hypothesis, there exists an L_2-coloring f_2 of G_2; by the choice of L_2, f_2 adds at most one new, allowed, flaw to the cutset $\{q_m, b_j, p_{m+1}\}$ (at p_{m+1}, on σ). Therefore $f_1' \cup f_2 \cup j$ is an L-coloring of G.

Case C.2.d.ii. $p_m \neq p_{m+1}$ and $s_{t-1}p_{m+1}$ is a flaw in j. Then, since $p_{m+1} \neq p_m$, $|J|(p_{m+1}) \succeq (1, 1)$. Let $\sigma' \in J^*(p_{m+1}) - \sigma$. Clearly p_{m+1} can be recolored with σ', if desired, to obtain a different J-coloring j' of S.

Let L_2 restrict L_2' by removing all colors not in $\{\sigma, \sigma'\}$ at p_{m+1}, and protecting both of $\{\sigma, \sigma'\}$ at p_{m+1}. Then (G_2, L_2) satisfies *C, and so an L_2-coloring f_2 of G_2 exists. By the choice of L_2, f_2 adds no new flaws to the cutset $\{q_m, b_j, p_{m+1}\}$ and is compatible with one of $\{j, j'\}$. Therefore either $f_1' \cup f_2 \cup j$ or $f_1' \cup f_2 \cup j'$ is an L-coloring of G. ■

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